IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

ADVANCED M-SERIES A GENERALIZED FUNCTION OF FRACTIONAL CALCULUS

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ABSTRACT

In recent year's many special functions given by mathematicians, here a new function termed as Advanced M-series function has been introduced. This Function is a particular case of H-function [1]. This function is important because hypergeometric function and Mittag-Leffler function follow as particular cases and these functions have great significance in the context of problems in physics, biology, engineering and applied sciences. It is to be noted that Mittag-Leffler [4,5] function occurs as solution of fractional integral equations in those subjects. In this paper we have also obtained the fractional integration and fractional differentiation of Advanced M-series function.

Mathematics Subject Classification: 33C60, 33E12, 82C31, 26A33.

KEYWORDS: Fractional Calculus, Advanced M-series and Riemann-Liouville Operator.

INTRODUCTION

THE ADVANCED M-SERIES

The **Advanced** M-series with $p+2$ upper parameters $a_{1,}a_{2,}\dots a_p, \gamma, \mu$and $q+1$ lower parameters $b_{1,}b_{2,}\dots b_q, \delta$ is

$$ {}_p\overset{\alpha,\beta}{M}_q\,(a_1\ .\ .\ .a_p,\gamma,\mu;,b_1\ .\ .\ .b_q,;z\ ) = {}_p\overset{\alpha,\beta}{M}_q\,(z) $$

$$ {}_p\overset{\alpha,\beta}{M}_q\,(z) = \sum_{k=o}^{\infty}\frac{(a_1)_k\ .\ .\ .(a_p)_k\,(\gamma)_k(\mu)_k}{(b_1)_k\ .\ .\ .(b_q)_k(\delta)_k(n_k)!\,(k)!}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{1.1} $$

Here,$\alpha,\beta\epsilon C,\ R(\alpha)>0, m>0$ and $(a_j)_k\ (b_j)_k\ ,(\gamma)_k(\mu)_k,(\delta)_k$are pochammer symbols. $(n_k)>0$The series (1.1) is defined when none of the denominator parameters $b_j s,\ j=1,2,\dots q$ is a negative integer or zero. If any parameter a_j is negative then the series (1.1))terminates into a polynomial in z. By using ratio test, it is evident that the series (1.1) is convergent for all z, when $q\geq p$, it is convergent for $|z|<1$ when $p=q+1$, divergent when $p>q+1$. In some cases the series is convergent for $z=1, z=-1$. Let us consider take,

$$ \beta=\sum_{j=1}^{p}a_j-\sum_{j=1}^{q}b_j $$

when $p=q+1$, the series is absolutely convergent for $|z|=1$ if $R(\beta)<0$, convergent for $z=-1$, if $0\leq R(\beta)<1$ and divergent for $|z|=1$, if $1\leq R(\beta)$.

Some Special Cases

A) If we put $(\delta)_k=(\mu)_k, n_k=1$ in equation (1.2)it convertes in k Function[7]

$$ {}_pk_q^{\alpha,\beta,\gamma}\,(a_1\ \dots a_p;,b_1\dots\ b_q;z\ ) = \sum_{k=o}^{\infty}\frac{(a_1)_k\ \dots\ (a_p)_k}{(b_1)_k\ \dots(b_q)_k}\frac{(\gamma)_k\,z^k}{(k)!\,\Gamma(\alpha k+\beta)} \tag{1.2} $$

B) If we put$(\delta)_k = (\mu)_k, n_k = 1, \gamma = 1$ in equation (1.2)it convertes in, Generalized M-Series [9]

$$ {}_p\overset{\alpha,\beta}{M}_q(z) = \sum_{k=o}^{\infty} \frac{(a_1)_k \ \ldots (a_p)_k}{(b_1)_k \ \ldots (b_q)_k} \quad (1.3) $$

C) If we put $(\delta)_k = (\mu)_k, n_k = 1, \gamma = 1, \beta = 1$ in equation (1.2)it convertes in, M-Series [8]

$$ {}_p\overset{\alpha}{M}_q(z) = \sum_{k=o}^{\infty} \frac{(a_1)_k \ \ldots (a_p)_k}{(b_1)_k \ \ldots (b_q)_k} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (1.4) $$

$D)$ ${}_0\overset{\alpha,\beta}{M}_0$ i.e. no p upper or q lower parameters and $(\delta)_k = (\mu)_k, n_k = 1$

$$ {}_p\overset{\alpha,\beta}{M}_q(\ldots;\ldots;z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k (z)^k}{\Gamma(\alpha k + \beta)(k)!} \quad (1.5) $$

Thus the series reduced to the Mittag-Leffler function as in [4,5]

MATHEMATICAL PREREQSITIES

The Riemann-Liouville fractional integral of order $v \in C$ is defined by Miller and Ross[3] (1993, p.45)

$$ {}_0D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-u)^{v-1} f(u) du, \quad (2.1) $$

where Re(v)>0. Following Samko et al. [6](1993, p. 37) we define the fractional derivative for $\alpha > 0$ in the form

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u)du}{(t-u)^{\alpha-n+1}}, \qquad (n = [\mathrm{Re}(\alpha)] + 1), \quad (2.2) $$

Where $[\mathrm{Re}(\alpha)]$ means the integral part of $\mathrm{Re}(\alpha)$.

FRACTIONAL INTEGRAL AND FRACTIONAL DERIVATIVE OF THE ADVANCED M-SERIES

Let us consider the fractional Riemann-Liouville (R-L) integral operator (for lower limit $a = 0$ with respect to variable z) of the Advanced M-Series (1.1).

$$ I_z^v \, {}_p\overset{\alpha,\beta}{M}_q(z) = \frac{1}{\Gamma(v)} \int_0^z (z-t)^{v-1} \, {}_p\overset{\alpha,\beta}{M}_q(t) \, dt $$

$$= \frac{1}{\Gamma(v)} \int_0^z (z-t)^{v-1} \sum_{k=o}^{\infty} \frac{(a_1)_k \ . \ . \ . (a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ . \ . \ . (b_q)_k (\delta)_k (n_k)! \, (k)!} \frac{t^k}{\Gamma(\alpha k+\beta)} \, dt$$

$$= \frac{1}{\Gamma(v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ . \ . \ . (a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ . \ . \ . (b_q)_k (\delta)_k (n_k)! \, (k)!} \frac{1}{\Gamma(\alpha k+\beta)} \int_0^z (z-t)^{v-1} t^k \, dt$$

$$= \frac{1}{\Gamma(v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ . \ . \ . (a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ . \ . \ . (b_q)_k (\delta)_k (n_k)! \, (k)!} \frac{1}{\Gamma(\alpha k+\beta)} z^{k+1+v-1} B(k+1, v)$$

$$= \frac{1}{\Gamma(v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ . \ . \ . (a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ . \ . \ . (b_q)_k (\delta)_k (n_k)! \, (k)!} \frac{1}{\Gamma(\alpha k+\beta)} z^{k+v} \frac{\Gamma(k+1)\Gamma(v)}{\Gamma(k+1+v)}$$

$$= z^v \sum_{k=o}^{\infty} \frac{(a_1)_k \ . \ . \ . (a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ . \ . \ . (b_q)_k (\delta)_k (n_k)! \, (k)!} \frac{z^k}{\Gamma(\alpha k+\beta)} \frac{\Gamma(k+1)}{\Gamma(k+1+v)}$$

$$= \frac{1}{\Gamma(1+v)} z^v \sum_{k=o}^{\infty} \frac{(a_1)_k \ . \ . \ . (a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ . \ . \ . (b_q)_k (\delta)_k (n_k)! \, (k)!} \frac{(1)_k}{(1+v)_k} \frac{z^k}{\Gamma(\alpha k+\beta)}$$

$$I_z^v \ {}_p\overset{\alpha,\beta}{M}_q \ (z) = \frac{1}{\Gamma(1+v)} z^v \ {}_pM_q^{\alpha,\beta} (a_1 \ldots a_p, \gamma, \mu, 1; b_1 \ldots b_q, \delta, 1+v; \ z \ ) \qquad (3.2)$$

$R-L$ fractional integral of Advanced M-Series where indices $p+2, q+1$ are increased to $(p+3)(q+2)$.

Analogously, $R-L$ fractional derivative operator of the Advanced M-Series with respect to z.

$$D_z^v \ {}_p\overset{\alpha,\beta}{M}_q \ (z) = \frac{1}{\Gamma(\mathrm{n}-v)} \left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-v-1} \ {}_pM_q^{\alpha,\beta} \ (t) \ dt$$

$$D_z^v \ {}_p\overset{\alpha,\beta}{M}_q \ (z) = \frac{1}{\Gamma(\mathrm{n}-v)} \left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-v-1} \sum_{k=o}^{\infty} \frac{(a_1)_k \ . \ . \ . (a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ . \ . \ . (b_q)_k (\delta)_k (n_k)! \, (k)!} \frac{t^k}{\Gamma(\alpha k+\beta)} dt$$

$$= \frac{1}{\Gamma(\mathrm{n}-v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ . \ . \ . (a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ . \ . \ . (b_q)_k (\delta)_k (n_k)! \, (k)!} \frac{1}{\Gamma(\alpha k+\beta)} \left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-v-1} \ t^k \ dt$$

$$= \frac{1}{\Gamma(\mathrm{n}-v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \ . \ . \ . (a_p)_k (\gamma)_k (\mu)_k}{(b_1)_k \ . \ . \ . (b_q)_k (\delta)_k (n_k)! \, (k)!} \frac{1}{\Gamma(\alpha k+\beta)} \left(\frac{d}{dz}\right)^n z^{k+n-v} \ B(k+1, n-v)$$

We use the modified Beta-function in above equation, which is defined as:

$$\int_a^b (b-t)^{\beta-1}\ (t-a)^{\alpha-1}\,dt\ = (b-a)^{\alpha+\beta-1}\,B(\alpha,\beta),$$

for $R(\alpha) > 0, R(\beta) > 0$

Again,

$$D_z^v\ {}_pM_q^{\alpha,\beta}\ (z) = \frac{1}{\Gamma(\mathrm{n}-v)}\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots(a_p)_k(\gamma)_k(\mu)_k}{(b_1)_k\ \ldots(b_q)_k(\delta)_k(n_k)!\,(k)!}\frac{1}{\Gamma(\alpha k+\beta)}\left(\frac{d}{dz}\right)^n$$

$$z^{k+n-v}\frac{\Gamma(k+1)\Gamma(\mathrm{n}-v)}{\Gamma(k+1+n-v)} \quad (3.3)$$

Where $k+1>0, n-v>0$

Differentiation n times the term z^{k+n-v} and using again $\Gamma(a+k) = (a)_k\Gamma(a)$, representation(3.3) reduces to

$$= \sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots(a_p)_k(\gamma)_k(\mu)_k}{(b_1)_k\ \ldots(b_q)_k(\delta)_k(n_k)!\,(k)!}\frac{\Gamma(k+n-v+1)}{\Gamma(\alpha k+\beta)\Gamma(k-v+1)}z^{k-v}\frac{\Gamma(k+1)}{\Gamma(k+1+n-v)}$$

$$= z^{-v}\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots(a_p)_k(\gamma)_k(\mu)_k}{(b_1)_k\ \ldots(b_q)_k(\delta)_k(n_k)!\,(k)!\,\Gamma(k-v+1)}z^k\frac{\Gamma(k+1)}{\Gamma(\alpha k+\beta)}$$

$$= \frac{1}{\Gamma(1-v)}z^{-v}\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ldots(a_p)_k(\gamma)_k(\mu)_k}{(b_1)_k\ \ldots(b_q)_k(\delta)_k(n_k)!\,(k)!}\frac{(1)_k}{\Gamma(\alpha k+\beta)(1-v)_k}\ z^k$$

$$D_z^v\ {}_pM_q^{\alpha,\beta}\ (z) = \frac{1}{\Gamma(1-v)}z^{-v}\ {}_pM_q^{\alpha,\beta}\,(a_1\ \ldots a_p,\gamma,\mu,1;b_1\ \ldots b_q,\delta,1-v;\ z\ ) \qquad (3.4)$$

$(k+1)>0$, gives a $R-L$ fractional derivative of Advanced M-series , where indices p, q are increased to (p+1),(q+1).

REFERENCES

[1] Fox, C.: The G and H-functions as symmetrical Fourier kernels. Trans. Amer. Math. Soc., 98 (1961), 395-429.

[2] Mathai, A. M. and Saxena, R. K.: The H-function with Applications in Statistics and other Disciplines. John Wiley and Sons. Inc. (1978), New York.

[3] Miller, K. S. and Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley and Sons. Inc.(1993), New York.

[4] Mittag- Leffler, Sur la nouvelle function $E_\alpha(x)$. C.R. Acad. Sci., Paris (Ser. II) 137 (1903), 554-558.

[5] Prabhakar, T.R. : A singular integral equation with generalized Mittag- Leffler function in the kernel, Yokohama Mathematical Journal 19 (1971), 7-15.

[6] Samko, S., Kilbas, A. Marichev, O.,: Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach, New York (1993).
[7] Sharma, K. and Dhakar, V. S.: Some Results Concerned to the Generalized Functions of Fractional Calculus Elixir Appl. Math. 51 (2012) 10961-10962.
[8] Sharma, M.: Fractional Integration and Fractional Differentiation of the M-Series FCAA Vol. 11, No. 2 (2008), 187-191.
[9] Sharma, M. And Jain, R.: A Note on a Generalized M-Series as a Special Function of Fractional Calculus, Fract. Calc. Appl. Anal. 12, No.4 (2009) 449-452.